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J. Math. Anal. Appl. 339 (2008) 1264–1276

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Bounds for effective strains of geometrically linear elastic multiwell model

Qi Tang [a], Kewei Zhang [b,*]

[a] *Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9RF, United Kingdom*
[b] *Department of Mathematics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, United Kingdom*

Received 13 September 2005

Available online 28 July 2007

Submitted by K.A. Lurie

Abstract

We give a direct estimate for the quasiconvex hull $Q^e(K)$ on linear strains generated by a finite set of linear strains $K \subset M_s^3$ of symmetric matrices. The problem is directly related to the microstructure modelled by the multiwell problem and its corresponding macroscopic effect. We bound the quasiconvex envelope $Q_e \operatorname{dist}(e(X), K)$ near an exposed edge of the convex hull $C(K)$ that does not have compatible connections. Our bounds depend on the weak type-(1, 1) estimate for certain singular integral operators and the geometric properties of the exposed edge.

Keywords: Effective strains; Geometrically linear elasticity; Multiwell model; Singular integral operators; Quasiconvex hull

In this paper we give two types of estimates for the quasiconvex hull $Q^e(K)$ of a finite set K of linear elastic strains in the three-dimensional geometrically linear elasticity. For a mapping $u \subset \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ with gradient Du, we denote the linear elastic strain of u by $e(Du) = (Du + (Du)^T)/2$. Throughout this paper, M_s^3 and $(M_s^3)^\perp \subset M^{3\times 3}$ are the mutually orthogonal linear subspaces of symmetric and skew-symmetric matrices in the space of real 3×3 matrices, respectively. Let $P_{M_s^3}$ and $P_{(M_s^3)^\perp}$ be the orthogonal projections to M_s^3 and $(M_s^3)^\perp$ respectively. It is easy to see that for any $X \in M^3$, $P_{M_s^3}(X) = \frac{1}{2}(X + X^T) = e(X)$.

In the geometrically linear theory of material microstructure and phase transformation, the multiwell model [5,12,13,15] has been extensively used (see [5] and references therein).

Let $K = \{A_1, A_2, \dots, A_m\} \subset M_s^n$ be a finite set of linear strains which are called linear elastic wells. Consider the multiwell energy density

$$W\big(e(X)\big) = \min_{1\leqslant i\leqslant m}\big[\big(\mathcal{M}_i\big(e(X) - A_i\big), e(X) - A_i\big) + \delta_i\big],$$

* Corresponding author.
E-mail addresses: q.tang@sussex.ac.uk (Q. Tang), k.zhang@swansea.ac.uk (K. Zhang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.07.051

where $\mathcal{M}_i$'s are positive definite four tensors called elastic moduli, $(\mathcal{M}_i(A), A)$ are positive definite quadratic forms on M_s^n and $\delta_i > 0$ are small positive numbers depending on the temperature. Denote by $I(u) = \int_\Omega W(e(Du))\,dx$ the total energy.

Let $u_j : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a sequence of mappings such that

$$I(u_j) \to \inf_{1\leqslant i\leqslant m} \delta_i \operatorname{meas}(\Omega), \quad \text{subject to certain boundary conditions.}$$

Since in general, W is not quasiconvex [5,7,11], the sequence $e(Du_j)$ is forced to oscillate among certain wells in K. The oscillation (or mathematically, the gradient Young measure) generated by the sequence $e(Du_j)$ models the microstructure of the material.

A simplified model [7] is as follows. Let $W(e(X)) = \operatorname{dist}^2(e(X), K)$, where $\operatorname{dist}^2(\cdot, K)$ is the squared distance function to K. In this case one would like to know how the sequence oscillates when

$$\lim_{j\to\infty} \int_\Omega \operatorname{dist}^2\big(e(Du_j), K\big)\,dx \to 0.$$

A closely related problem is to find the set of all 'effective linear strains' generated by K, that is, the smallest compact set call the quasiconvex hull $Q^e(K)$ containing K which satisfies that if $\operatorname{dist}^2(e(Du_j), K) \to 0$ in L^2 and converges weakly to $e(Du)$, then $e(Du(x)) \in Q^e(K)$ almost everywhere. One uses $Q^e(K)$ to locate the average of microstructures or effective strains. Although there are some examples of explicit calculation of quasiconvex hulls for finite sets of linear strains [7,12], it is not known how to calculate $Q^e(K)$ even for a three point set $K \subset M_s^3$ in general [7].

A symmetric matrix $A \in M^3$ is called a compatible linear strain, by its algebraic definition if either A is of rank-one or $\operatorname{rank}(A) = 2$ and the two non-zero eigenvalues of A have opposite signs [12]. Otherwise, A is called incompatible. Let $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ be the ordered eigenvalues of A, clearly, A is compatible if and only if $\lambda_2 = 0$. Alternatively, the geometric definition is that there is some rank-one matrix $A_0 \in M^{3\times 3}$ such that $A = P_{M_s^3}$. Let $E_0 = \operatorname{span}[A] \subset M_s^3$ be the one-dimensional subspace spanned by A, then A is incompatible if and only if the subspace $E_1 = (M_s^3)^\perp \oplus E_0 \subset M^{3\times 3}$ does not have rank-one matrices.

We say that a finite set $K \subset M_s^3$ has a compatible connection if there are two points $A_1, A_2 \in K$ such that $A_1 - A_2$ is compatible. Clearly, a sufficient condition for $Q^e(K) \neq K$ is that K has compatible connections [7]. However, this is not a necessary condition for non-trivial quasiconvex hull as there is an example in [7] (also see Example 1 below) showing that there is a three point set $K = \{A, B, C\} \subset M_s^3$ such that K does not have compatible connections while $Q^e(K) \neq K$.

An approximation of $Q^e(K)$ from inside, called the lamination convex hull $L_c^e(K)$ of K has been used extensively in the material sciences literature (see [5,13] and references therein). A refined version by using laminates within laminates [7] sometimes gives a better approximation. However, it is not known when these inner hulls equal $Q^e(K)$ except the simple case of a two point set [12] and certain finite sets in M_s^2 [7].

A qualitative result known as the equal hull property for compact sets of linear strains was obtained in [24] which that $Q^e(K) = C(K)$ if and only if $L_c^e(K) = C(K)$ where $L_c^e(K)$ and $C(K)$ are the closed lamination convex hull of K on linear strains and the convex hull of K respectively. It was observed in [24] that if $Q^e(K)$ is not convex, the 'non-convexity' must occur near the boundary $\partial C(K)$. The example in [7] (see Example 1 below) suggests that to bound the quasiconvex hull $Q(K)$ from outside with limited information that K is finite and $Q^e(K)$ is not convex, one has to bound $Q^e(K)$ near $\partial C(K)$.

We give two different quantitative estimates of $Q^e(K)$ in this paper.

(i) We first establish a lower bound of the quasiconvex envelope $Q\operatorname{dist}(e(X), K)$ of the distance function near an exposed edge (1-face) L of $C(K)$ that is not compatible. Intuitively, suppose $\dim C(K) = 3$ and the exposed face L is one-dimensional, then along L we can chip a wedge like slice off $C(K)$ without touching $Q^e(K)$. The advantage of this approach is that the estimates are independent of the 'size' or diameter of the set K as we only need the information of the relative position to certain planes associated with the 1-face L of $C(K)$. The disadvantage is that certain constants related to some singular integral operators are difficult to be made explicit.

(ii) Our second approach is based on certain explicitly defined rank-one convex quadratic functions defined on the space of linear elastic strains. This class of functions are well understood [26]. The estimates can be explicit which depend on the geometry of the set K and the ellipticity constant of the 1-face L mentioned above. However, such an estimate does depend on the size of K.

We focus on approach (i) first. From the structure of convex polytopes [6,18], we see that a k-dimensional polytope P is the convex hull of all of its exposed edges. Therefore, if $Q^e(K) \neq C(K)$ and K is finite, we may claim that there is at least one 1-face (i.e., an exposed edge) which does not have compatible connections [24]. If all the exposed edges are compatible, then $L^e(K) = C(K)$ (see [5]) hence $Q^e(K)$ is trivial.

From now on, we denote by $K \subset M_s^3$ a non-empty finite set with $Q^e(K) \neq C(K)$. Let $d = \dim C(K)$ be the affine dimension of the convex hull $C(K)$ and we assume that $d > 1$ as the case $d = 1$ is trivial. Let $M \subset M_s^3$ be the plane containing $C(K)$ with $\dim M = d$. For a point $X \in M_s^3$ and a set $V \subset M_s^3$, let $V - X = \{Y - X,\ Y \in V\}$.

Suppose $L \subset C(K)$ is a non-trivial exposed edge of polytope $C(K)$ which does not have compatible connections. Let E be the plane that contains $K_0 = L \cap K$ with $\dim(L) = \dim(E)$. Since it is well known and easy to check by definition that quasiconvex hull is translation invariant in the sense that $Q^e(K - X) = Q^e(K) - X$. We may assume that $0 \in K_0$ hence E is a linear subspace of M_s^3.

Let W be a supporting plane of $C(K)$ such that $L = C(K) \cap W$. Obviously, $E \subset W \subset M$, hence both W and M are subspaces and $\dim W = d - 1$. We denote by V be the orthogonal complement of E in W. Denote by e the unit normal vector (a matrix with norm 1) of W in M pointing to the half-space of M containing $C(K)$. Let $F = \mathrm{span}[e] \oplus V$ and define

$$\cos\theta_W = \inf\left\{\frac{e \cdot X}{|P_F(X)|},\ X \in K \setminus E\right\}, \tag{1}$$

where P_F is the orthogonal projection from M_s^3 to F. Since K is finite, we have $\cos\theta_W > 0$ and

$$e \cdot X \geqslant \cos\theta_W \left|P_F(X)\right|$$

for all $X \in K$. The above inequality still hold if $X \in K$ and $P_F(X) = 0$ because in this case $X \perp e$.

Next we may optimize the angle θ_W by varying the possible supporting planes W satisfying $W \cap C(K) = L$. Let $\mathcal{W}$ be the collection of supporting planes of $C(K)$ such that $L = W \cap C(K)$. Then the optimal angle $\theta_0 \in (0, \pi/2)$ is defined as

$$\cos\theta_0 = \sup\{\cos\theta_W,\ W \in \mathcal{W}\}.$$

Since K is finite and M is a finite-dimensional plane, it is easy to see that $\cos\theta_0$ can be reached by some W^* with $W_0^* = E \oplus V^*$ and $F^* = \mathrm{span}[e] \oplus V^*$.

From now on we drop the superscript $*$ and denote by W the optimal supporting plane and W, F, V the corresponding subspaces given above. Thus we have, for $X \in K$,

$$e \cdot X \geqslant \cos\theta_0 \left|P_F(X)\right|. \tag{2}$$

Note that E is the subspace generated by the exposed edge L which contains 0. In general, E itself is not a supporting plane of $C(K)$ unless $d = 2$. In that case, $E = W$ and $\theta_0 = 0$, hence $\cos\theta_0 = 1$. Also note that θ_0 is the angle which makes $\cos(\theta_0)$ the largest among all $\cos(\theta_W)$.

The following is the main result obtained by approach (i).

Theorem 1. *Let $K \subset M_s^3$ be a finite set such that $C(K) \neq Q^e(K)$. Assume that $d = \dim(C(K))$ with $1 < d \leqslant 6$ and let $L \subset C(K)$ be an exposed edge of $C(K)$ without compatible connections. Suppose $0 \in L$ and $E = \mathrm{span}[L]$. Then*

$$Q_e \operatorname{dist}(X, K) \geqslant C(E, \theta_0, \sigma)\left(\operatorname{dist}(X, K) - (1 + \sigma) C_E \frac{1 + \cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right|\right), \tag{3}$$

for any $\sigma > 0$, where $C(E, \theta_0, \sigma) > 0$ is a constant given by (15) *below, $P_{E^\perp}$ is the orthogonal projection from M_s^3 to the orthogonal complement of E in M_s^3 and C_E is the constant given by* (5) *below. Furthermore,*

$$\operatorname{dist}(X, K) > C_E \frac{1 + \cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right| \quad \textit{implies} \quad X \notin Q^e(K). \tag{4}$$

Remark 1. It was established in [26] that in M_s^3, the largest dimension of subspaces without compatible matrices is one. Therefore in Theorem 1, the only non-trivial exposed faces of $C(K)$ without compatible connections are exposed edges. It is also easy to see that if all exposed edges have compatible connections, the quasiconvex hull $Q^e(K)$ on linear strains must be trivial, that is $Q^e(K) = C(K)$ [5].

If we apply Theorem 1 to each non-trivial exposed edges of $C(K)$ without compatible connections, we may define a non-convex set which stays between $Q^e(K)$ and $C(K)$.

Remark 2. If we do not assume that $0 \in L$ and suppose $X_0 \in L$, then (4) can be written in this general case as

$$\operatorname{dist}(X, K) > C_{E_0} \frac{1+\cos\theta_0}{\cos\theta_0} \left|P_{E_0^\perp}(X - X_0)\right| \quad \text{implies} \quad X \notin Q^e(K),$$

where $E_0 = \operatorname{span}[L - X_0]$.

Theorem 2. *Let $E = \operatorname{span}[A_0] \subset M_s^3$ be a one-dimensional subspace such that $|A_0| = 1$ and A_0 is not a compatible strain. Let $\lambda_1 < \lambda_2 < \lambda_3$ be the eigenvalues of A_0. Then there is a constant $C_E > 0$ in the form*

$$C_E = C\left(1 + \frac{1}{\lambda_2^6}\right), \tag{5}$$

where $C > 0$ is an absolute constant, such that for every $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$,

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^3,\ \left|e\left(D\phi(x)\right)\right| \geqslant \lambda\right\}\right) \leqslant \frac{C_E}{\lambda} \int_{\mathbb{R}^3} \left|P_{E^\perp}\left(e\left(D\phi(y)\right)\right)\right| dy, \quad \textit{for all } \lambda > 0. \tag{6}$$

Remark 3. We would like to make some comments on how singular integral operators are related to quasiconvex hulls in the calculus of variations. Applications of powerful harmonic analysis methods to vectorial calculus of variations go back to E. Acerbi and N. Fusco in their seminal paper [2] on quasiconvexity and lower semicontinuity of variational integrals. In [2], they successfully modified a bounded $W^{1,p}$ sequence into a bounded $W^{1,\infty}$ sequence based on the fundamental work of F.C. Liu [14] on Luzin type theorems for Sobolev functions, where the maximal function estimates is essential. A further application along this direction was contained in [22] which leads to the work [23] that for a compact set $K \subset M^{N\times n}$, the notion of p-quasiconvex hulls $Q_p(K)$ coincides with the quasiconvex hull $Q(K)$ defined in [21]. In the context of linear elastic strains we consider in the present paper, Liu's work was generalized in [9]. The work [9] was based on singular integral operator estimates related to functions with bounded deformation [1]. However, due to the counter-example in [16] that even for bounded linear strains in L^∞, the gradient of the mapping might not be in L^∞, one can only show that $Q_p^e(K)$ is independent of $1 \leqslant p < \infty$ (see [27]) which will be our definition of $Q^e(K)$ later.

Based on the singular integral operator T obtained in Proposition 2 below, the study of the Beltrami operator on the complex plane $\mathbb{C}$ [3] and its applications to gradient Young measures supported in $M^{2\times 2}$, we can use T to define a class of Beltrami-like operators by using T and to study some general properties of the approximate sequences for gradient Young measures supported in certain 'quasi-regular' sets in M_s^3. We will briefly describe such applications in Remark 4 below.

We will present our results on approach (ii) after we establish Theorems 1 and 2.

After notation and preliminaries, we establish Theorem 1 first by accepting Theorem 2. We prove Theorem 2 afterwards followed by some comments on the singular integral operator involved and an example as an illustration of both Theorems 1 and 2. In the last part of this paper we take approach (ii) and give an alternative estimate of the quasiconvex hull $Q^e(K)$ near an exposed incompatible 1-face of $C(K)$.

When we consider linear elastic strains, we restrict ourselves to $M^{3\times 3}$ of 3×3 real matrices with $\mathbb{R}^9$ norm, and its subspace $M_s^3 \subset M^{3\times 3}$ of symmetric matrices. Let $\operatorname{dist}(A, K) = \inf_{P\in K} |A - P|$ be the distance function from a point $A \in \mathbb{R}^n$ to a set $K \subset \mathbb{R}^n$. Let Ω be a non-empty, open and bounded subset of $\mathbb{R}^n$. We denote by Du the gradient of a (vector-valued) function $u : \Omega \to \mathbb{R}^N$ and define the space $C_0^k(\Omega, \mathbb{R}^N)$ in the usual way.

A continuous function $f : M^{N\times 3n} \to \mathbb{R}$ is quasiconvex at $A \in M^{N\times n}$ if for any smooth function $\phi : \Omega \to \mathbb{R}^N$ compactly supported in Ω,

$$\int_\Omega f\big(A + D\phi(x)\big)\,dx \geqslant \int_\Omega f(A)\,dx.$$

If f is quasiconvex at every $A \in M^{N\times n}$, it is called a quasiconvex function [4,8,17]. The class of quasiconvex functions is independent of the choice of Ω [8].

For a given continuous function $f : M^{N\times n} \to \mathbb{R}$, the quasiconvex envelope $Q(f)$ is defined by

$$Q(f) = \sup\{g \leqslant f,\ g \text{ is quasiconvex}\},$$

and can be calculated by using the formula

$$Q(f)(A) = \inf_{\phi\in C_0^\infty(D,\mathbb{R}^N)} \int_D f\big(A + D\phi(x)\big)\,dx,$$

where $D \subset \mathbb{R}^n$ is the unit cube [8].

For a closed set $K \subset M^{N\times n}$, the quasiconvex hull $Q(K)$ is defined by [21]

$$Q(K) = \Big\{X \in M^{N\times n},\ f(X) \leqslant \sup_{Y\in K} f(Y),\ \text{for every quasiconvex } f : M^{N\times n} \to \mathbb{R}\Big\}.$$

When K is compact, the quasiconvex hull $Q(K)$ can be defined by a single quasiconvex function [23] as

$$Q(K) = \big\{A \in M^{N\times n},\ Q\,\mathrm{dist}^p(A, K) = 0\big\}$$

for each $1 \leqslant p < \infty$.

For a continuous function $f : M_s^3 \to \mathbb{R}$, we define the quasiconvex envelope $Q_e(f)$ on linear strain as

$$Q_e\big(f(A)\big) = \inf_{\phi\in C_0^\infty(D,\mathbb{R}^3)} \int_D f\big(A + e\big(D\phi(x)\big)\big)\,dx, \quad \text{for } A \in M_s^3.$$

We define the p-quasiconvex hull $Q^e(K)$ on linear strain for a compact set $K \subset M_s^3$ as

$$Q_p^e(K) = \big\{A \in M_s^3,\ Q_e\,\mathrm{dist}^p(A, K) = 0\big\}.$$

In fact, for each $p \in [1, +\infty)$, it was established in [27] that $Q_1^e(K) = Q_p^e(K)$ for all $1 \leqslant p < \infty$. Thus we define the quasiconvex hull on the set K of linear strains as $Q^e(K) = Q_1^e(K)$ hence $Q^e(K) = Q_p^e(K)$ for $1 \leqslant p < \infty$. Note that this definition of quasiconvex hull is weaker than that for compact sets in $M^{N\times n}$ as when we lift our sets $K \subset M_s^3$ back to $M^{3\times 3}$, the set $K \oplus (M_s^3)^\perp$ is an unbounded set (also see Remark 3).

The following result in [10] is a consequence of the *measurable selection lemma*.

Proposition 1. *Let $K \subset \mathbb{R}^n$ be a compact subset and let $u : \overline{\Omega} \to \mathbb{R}^n$ be a continuous mapping. Then there exists a measurable mapping $\tilde{u} : \Omega \to K$ such that for all $x \in \Omega$*

$$\big|u(x) - \tilde{u}(x)\big| = \mathrm{dist}\big(u(x), K\big).$$

We conclude our preparation by stating a standard result on the weak type-(1, 1) estimate for singular convolution operators [19,20] by considering the kernels of their Fourier transform.

Proposition 2. *Let $m(\cdot) : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a complex valued function such that*

$$\big|\partial_\xi^\alpha m(\xi)\big| \leqslant A_{|\alpha|}|\xi|^{-|\alpha|}, \quad \text{for } 0 \leqslant |\alpha| \leqslant l,$$

where l is the smallest integer $> n/2$, and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with length $|\alpha| = \sum_{i=1}^n \alpha_i$. Then there is a singular integral operator in the form

$$(Tf)(x) = cf(x) + p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

which is bounded from $L^2(\mathbb{R}^n)$ to itself. Furthermore, T satisfies the following weak type-$(1,1)$ estimate

$$\operatorname{meas}\bigl(\bigl\{x\in\mathbb{R}^n \bigm| (Tf)(x)>\lambda\bigr\}\bigr)\leqslant\frac{C_m}{\lambda}\int_{\mathbb{R}^n}\bigl|f(y)\bigr|\,dy,$$

for all $\lambda>0$, where $C_m>0$ is a constant in the form

$$C_m=C(n,l)\left(1+\sum_{|\alpha|}^{l}A_{|\alpha|}\right),$$

with $A_{|\alpha|}$'s given by the above.

Now we establish our main results. We prove Theorem 1 first, accepting Theorem 2 for the moment.

Proof of Theorem 1. Let $X\in M_s^3$ be fixed and let $D\subset\mathbb{R}^3$ be the unit cube. Suppose (ϕ_j) is a sequence in $C_0^\infty(D,\mathbb{R}^3)$ such that

$$\lim_{j\to\infty}\int_D \operatorname{dist}\bigl(X+e(D\phi_j),K\bigr)\,dx=Q_e\operatorname{dist}(X,K):=a\geqslant 0.$$

We extend ϕ_j to $\mathbb{R}^3$ by zero outside D so that $\phi_j\in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$.

Now we apply the measurable selection lemma (Proposition 1) to the function

$$F(x,Y)=\bigl|X+e\bigl(D\phi_j(x)\bigr)-Y\bigr|$$

for $x\in\bar{D}$ and $Y\in K$. There exists a measurable mapping $X_j:D\to K-X_0$, such that

$$\bigl|X+e\bigl(D\phi_j(x)\bigr)-X_j(x)\bigr|=\operatorname{dist}\bigl(X+e\bigl(D\phi_j(x)\bigr),K\bigr)$$

almost everywhere in Ω. From (2), we see that $Y_0\cdot X_j\geqslant\cos\theta_0|P_FX_j|$ a.e. in D. Let

$$\int_D \operatorname{dist}(X+D\phi_j,K)\,dx=a+\delta_j,$$

where $\delta_j\geqslant 0$ and $\lim_{j\to\infty}\delta_j=0$. Since ϕ_j is zero on the boundary of D, $\int_D e(D\phi_j)\,dx=0$. We have

$$\begin{aligned}a+\delta_j&=\int_D\bigl|X+e\bigl(D\phi_j(x)\bigr)-X_j(x)\bigr|\,dx\geqslant\int_D\bigl|Y_0\cdot\bigl(X+e\bigl(D\phi_j(x)\bigr)-X_j(x)\bigr)\bigr|\,dx\\&\geqslant\left|\int_D Y_0\cdot\bigl(X+D\phi_j(x)-X_j(x)\bigr)\,dx\right|\geqslant\int_D Y_0\cdot X_j(x)\,dx-|Y_0\cdot X|.\end{aligned}\tag{7}$$

From (2) we have

$$\int_D Y_0\cdot X_j(x)\,dx\geqslant\cos\theta_0\int_D|P_FX_j|\,dx.\tag{8}$$

Combining (7) and (8) we have

$$a+\delta_j\geqslant\cos\theta_0\int_D|P_FX_j|\,dx-|Y_0\cdot X|$$

so that

$$\int_D|P_FX_j|\,dx\leqslant\frac{1}{\cos\theta_0}\bigl(a+\delta_j+|Y_0\cdot X|\bigr).\tag{9}$$

On the other hand, note that $X_j(x) \in K \subset M$ a.e., and $E^\perp = F \oplus (M)^\perp$, hence $P_M^\perp(X_j) = 0$ a.e., where all of the orthogonal complements are in M_s^3. We have

$$\begin{aligned} a+\delta_j &= \int_D \operatorname{dist}\bigl(X + e(D\phi_j), K\bigr)\,dx \\ &= \int_D \bigl|X + e(D\phi_j) - X_j\bigr|\,dx \\ &\geqslant \int_D \bigl|P_{E^\perp}\bigl(X + e(D\phi_j) - X_j\bigr)\bigr|\,dx \\ &\geqslant \int_D \bigl|P_{E^\perp}\bigl(e(D\phi_j)\bigr)\bigr|\,dx - \bigl|P_{E^\perp}(X)\bigr| - \int_D |P_F X_j|\,dx. \end{aligned} \tag{10}$$

Therefore, by (9) and (10),

$$\begin{aligned} \int_D \bigl|P_{E^\perp}\bigl(e(D\phi_j)\bigr)\bigr|\,dx &\leqslant (a+\delta_j) + \bigl|P_{E^\perp}(X)\bigr| + \int_D |P_F X_j|\,dx \\ &\leqslant (a+\delta_j) + \bigl|P_{E^\perp}(X)\bigr| + \frac{1}{\cos\theta_0}\bigl(a+\delta_j + |e\cdot X|\bigr) \\ &\leqslant \frac{1+\cos\theta_0}{\cos\theta_0}\bigl(\bigl|P_{E^\perp}(X)\bigr| + a + \delta_j\bigr). \end{aligned} \tag{11}$$

From Theorem 2, we have

$$\operatorname{meas}\bigl(\bigl\{x \in \mathbb{R}^3,\ \bigl|e\bigl(D\phi_j(x)\bigr)\bigr| > \lambda\bigr\}\bigr) \leqslant \frac{C_E}{\lambda}\int_D \bigl|P_{E^\perp} e(D\phi_j)\bigr|\,dx \leqslant C_E \frac{1+\cos\theta_0}{\lambda\cos\theta_0}\bigl(\bigl|P_{E^\perp}(X)\bigr| + a + \delta_j\bigr),$$

for every $\lambda > 0$, where $C_E > 0$ is the constant given by Theorem 2. Since the distance function $\operatorname{dist}(\cdot, K)$ is Lipschitz satisfying

$$\bigl|\operatorname{dist}(A,K) - \operatorname{dist}(B,K)\bigr| \leqslant |A-B|$$

for $A, B \in M_s^3$, we see that

$$\operatorname{dist}(X,K) > \operatorname{dist}\bigl(X + e\bigl(D\phi_j(x)\bigr), K\bigr) + \lambda \quad \text{implies} \quad \bigl|e\bigl(D\phi_j(x)\bigr)\bigr| > \lambda.$$

In other words,

$$D_\lambda := \bigl\{x \in D,\ \operatorname{dist}(X,K) > \operatorname{dist}\bigl(X + e\bigl(D\phi_j(x)\bigr), K\bigr) + \lambda\bigr\} \subset \bigl\{x \in D,\ \bigl|e\bigl(D\phi_j(x)\bigr)\bigr| > \lambda\bigr\},$$

so that

$$\operatorname{meas}(D_\lambda) \leqslant C_E \frac{1+\cos\theta_0}{\lambda\cos\theta_0}\bigl(\bigl|P_{E^\perp}(X)\bigr| + a + \delta_j\bigr).$$

Choosing, for each fixed $\sigma > 0$,

$$\lambda = (1+\sigma) C_E \frac{1+\cos\theta_0}{\cos\theta_0}\bigl(\bigl|P_{E^\perp}(X)\bigr| + a\bigr), \tag{12}$$

we see that for sufficiently large $j > 0$, $\operatorname{meas}(\{x \in \mathbb{R}^3,\ |D\phi_j(x)| > \lambda\}) < 1$, so that

$$\begin{aligned} a+\delta_j &= \int_D \operatorname{dist}\bigl(X + e\bigl(D\phi_j(x)\bigr), K\bigr)\,dx \geqslant \int_{D\setminus D_\lambda} \operatorname{dist}\bigl(X + e\bigl(D\phi_j(x)\bigr), K\bigr)\,dx \\ &\geqslant \bigl[\operatorname{dist}(X,K) - \lambda\bigr]\biggl[1 - C(E)\frac{1+\cos\theta_0}{\lambda\cos\theta_0}\bigl(\bigl|P_{E^\perp}(X)\bigr| + a + \delta_j\bigr)\biggr], \end{aligned}$$

for sufficiently large $j > 0$. Passing to the limit in the above inequality, we obtain

$$a \geqslant \left[\operatorname{dist}(X, K) - \lambda\right]\left[1 - C_E \frac{1+\cos\theta_0}{\lambda\cos\theta_0}\left(\left|P_{E^\perp}(X)\right| + a\right)\right]. \tag{13}$$

Substituting (12) into (13), we have

$$a \geqslant \frac{1}{1+\sigma C_E \frac{1+\cos\theta_0}{\cos\theta_0}}\left(\frac{\sigma}{1+\sigma}\right)\left(\operatorname{dist}(X, K) - (1+\sigma)C_E \frac{1+\cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right|\right).$$

Thus

$$Q_e \operatorname{dist}(X, K) \geqslant C(E, \theta_0, \sigma)\left(\operatorname{dist}(X, K) - (1+\sigma)C_E \frac{1+\cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right|\right) \tag{14}$$

for each fixed $\sigma > 0$, where

$$C(E, \theta_0, \sigma) = \frac{1}{1+\sigma C_E \frac{1+\cos\theta_0}{\cos\theta_0}}\left(\frac{\sigma}{1+\sigma}\right). \tag{15}$$

Now if $X \in M_s^3$ satisfies

$$\operatorname{dist}(X, K) > C_E \frac{1+\cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right|, \tag{16}$$

there is some $\sigma > 0$ such that

$$\operatorname{dist}(X, K) - (1+\sigma)C_E \frac{1+\cos\theta_0}{\cos\theta_0}\left|P_{E^\perp}(X)\right| > 0,$$

hence from (14), $Q_e \operatorname{dist}(X, K) > 0$, which implies $X \notin Q^e(K)$. The proof is finished. □

Now we turn to the proof of Theorem 2. Suppose $E = \operatorname{span}[A_0]$ is a one-dimensional subspace generated by an incompatible strain $A_0 \in M_s^3$ with $|A_0| = 1$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ be the eigenvalues of A_0, we see that $\lambda_2 \neq 0$. Let

$$\lambda_+ = \max\{\lambda_3, 0\}, \qquad \lambda_- = \max\{-\lambda_1, 0\}.$$

It was established in [25] the following sharp inequalities that

$$\left(a^T A_0 b\right)^2 \leqslant \frac{\lambda_+^2 + \lambda_-^2}{2}\left(|a|^2|b|^2 + \left(a^T b\right)^2\right), \tag{17}$$

for all $a, b \in \mathbb{R}^3$ as column vectors. Also

$$|A_0 a|^2|a|^2 \leqslant \left\{\frac{\lambda_+^2 + \lambda_-^2}{2}\right\}|a|^4 + \frac{1}{2}(Xa, a)^2.$$

As a consequence, we have

$$\left|P_{E^\perp}\left(e(a \otimes b)\right)\right|^2 \geqslant \mu\left|P_E\left(e(a \otimes b)\right)\right|^2 \tag{18}$$

where $\mu = 1 - \lambda_+^2 - \lambda_-^2$. Clearly, $\mu \geqslant \lambda_2^2$.

Proof of Theorem 2. We first define a multiplier. Let the linear mapping $L(\xi) : M^3(\mathbb{C}) \to M^3(\mathbb{C})$ for each $\xi \in \mathbb{R}^3$ with $\xi \neq 0$ be defined as

$$L(\xi)(X) = \frac{G(\xi)(X) + (G(\xi)(X))^T}{2}, \tag{19}$$

where

$$G(\xi)(X) = \left\{\left[\frac{I + (\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2} - \left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]^{-1} X\left(\frac{\xi}{|\xi|}\right)\right\} \otimes \left(\frac{\xi}{|\xi|}\right). \tag{20}$$

Since $P_{E^\perp}(e(\eta\otimes\xi)) = e(\eta\otimes\xi) - (\xi^T E\eta)E$, we see that

$$\begin{aligned}
&G(\xi)\big(P_{E^\perp}(\eta\otimes\xi)\big)\\
&=\left\{\left[\frac{I+(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2}-\left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]^{-1}\big(e(\eta\otimes\xi)-\big(\xi^T E\eta\big)E\big)\left(\frac{\xi}{|\xi|}\right)\right\}\otimes\left(\frac{\xi}{|\xi|}\right)\\
&=\left\{\left[\frac{I+(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2}-\left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]^{-1}\left[\frac{I+(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2}-\left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]\eta\right\}\otimes\xi\\
&=\eta\otimes\xi,
\end{aligned}$$

hence

$$L(\xi)\big(P_{E^\perp}\big(e(\eta\otimes\xi)\big)\big) = e(\eta\otimes\xi).$$

We can calculate $G(\xi)(\cdot)$ explicitly by using simple linear algebra. Firstly, we have

$$\begin{aligned}
&\left[\frac{I+(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2}-\left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]^{-1}\\
&=\left(2I-\left(\frac{\xi}{|\xi|}\right)\left(\frac{\xi}{|\xi|}\right)^T\right)+\frac{[(2I-(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T)(E(\frac{\xi}{|\xi|}))][(2I-(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T)(E(\frac{\xi}{|\xi|}))]^T}{1-(E(\frac{\xi}{|\xi|}))^T(2I-(\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T)(E(\frac{\xi}{|\xi|}))}.
\end{aligned}$$

Clearly G is homogeneous of degree 0 in ξ and if we write $X\in M_c^{3\times 3}$ as a complex vector in $\mathbb{C}^9$, there is an 9×9 matrix $\mathcal{M}(\xi)$ such that

$$G(\xi)(X) = \mathcal{M}(\xi)X,$$

with the right-hand side of the above equality the product of a matrix and a vector. We only need to give a weak type-(1, 1) estimate of the operator defined by the multiplier $L(\xi)$.

Now if we define an operator V from $L^2(M^{3\times 3})$ to $L^2(M^{3\times 3})$ by its Fourier transform

$$\widehat{Vf}(\xi) := G(\xi)\big(\hat{f}(\xi)\big),$$

we see that [19] T is bounded with $V(P_{E^\perp}(e(Du))) = Du$ for any $u\in W^{1,2}(\mathbb{R}^3,\mathbb{R}^3)$. Since the components of $G(\xi)$ are all of C^∞ in $\mathbb{R}^3\setminus\{0\}$,

$$\left|\partial_{\xi_\beta}\left(\frac{\xi_\alpha}{|\xi|}\right)\right|\leqslant\frac{2}{|\xi|},$$

and by (18)

$$\begin{aligned}
1-\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\left(2I-\left(\frac{\xi}{|\xi|}\right)\left(\frac{\xi}{|\xi|}\right)^T\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)&=1-\left[2\left|E\left(\frac{\xi}{|\xi|}\right)\right|^2-\left(\left(E\left(\frac{\xi}{|\xi|}\right)\right)\cdot\left(\frac{\xi}{|\xi|}\right)\right)^2\right]\\
&\geqslant 1-\big(\lambda_+^2+\lambda_-^2\big)\geqslant\lambda_2^2,
\end{aligned}$$

we see that each component of $G(\xi)(E_{i\alpha})$ satisfies

$$\big|\partial_{\xi^\tau}\big(G(\xi)(E_{i\alpha})\big)_{j\beta}\big|\leqslant C_{|\tau|}\left(1+\frac{1}{\lambda_2^{2+2|\tau|}}\right),$$

for all non-negative multi-indices $\tau=(\tau_1,\tau_2,\tau_3)$ with $|\tau|=\sum_{i=1}^3\tau_i$, $\partial_{\xi^\tau}=\partial\xi_1^{\tau_1}\partial\xi_2^{\tau_2}\partial\xi_3^{\tau_3}$, where $E_{i\alpha}$ is the 3×3 matrices with (i,α) entry 1 and zero otherwise. Thus the conditions for [20, pp. 246–247, Proposition 2(b)], are satisfied for $l=2=|\tau|>3/2$. Hence the weak-(1, 1) estimate for V holds [19, p. 29, p. 34]:

$$\operatorname{meas}\big(\big\{x\in\mathbb{R}^n,\ \big|(Vf)(x)\big|\geqslant\lambda\big\}\big)\leqslant C_1\left(1+\frac{1}{\lambda_2^6}\right)\frac{1}{\lambda}\int_{\mathbb{R}^3}|f|\,dx,$$

where $C_1 > 0$ is an absolute constant. Also, for the operator defined by $(G(\xi)(X))^T$, we have a similar estimate. Thus if we let T be the operator corresponding to the multiplier $L(\xi)$, we have the weak type-(1, 1) estimate

$$\operatorname{meas}\bigl(\bigl\{x \in \mathbb{R}^3,\ \bigl|(Tf)(x)\bigr| \geqslant \lambda\bigr\}\bigr) \leqslant \frac{C_E}{\lambda} \int_{\mathbb{R}^3} |f|\,dx,$$

with

$$C_E = C\left(1 + \frac{1}{\lambda_2^6}\right), \tag{21}$$

where $C > 0$ is an absolute constant. In particular, if $f = P_{E^\perp}(e(Du))$ for $u \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $T(P_{E^\perp}(e(Du))) = e(Du)$, hence

$$\operatorname{meas}\bigl(\bigl\{x \in \mathbb{R}^3,\ \bigl|e\bigl(Du(x)\bigr)\bigr| \geqslant \lambda\bigr\}\bigr) \leqslant \frac{C_E}{\lambda} \int_{\mathbb{R}^3} \bigl|P_{E^\perp}\bigl(e(Du)\bigr)\bigr|\,dx. \qquad \square$$

Remark 4. Our proof of Theorem 2 follows the approach in [19,20] and is unlikely to be sharp. However, we believe that this is the first quantitative estimate of the effective strains under very limited information. We do not know whether different approaches using the multiplier in the proof can produce a sharper and more explicit bound in our weak-(1, 1) estimate.

Let us look at the singular integral operator T defined in Proposition 2 more closely. The Fourier transform side of T is given by the linear mapping $L(\xi)(X)$ defined by (19) with $G(\xi)(X)$ given by (20). We remark here that the constant c in the definition of Tf is zero as we can write in (20)

$$G(\xi)(X) = H(\xi)(X) \otimes \frac{\xi}{|\xi|}$$

where

$$H(\xi)(X) = \left[\frac{I + (\frac{\xi}{|\xi|})(\frac{\xi}{|\xi|})^T}{2} - \left(E\left(\frac{\xi}{|\xi|}\right)\right)\left(E\left(\frac{\xi}{|\xi|}\right)\right)^T\right]^{-1} X\left(\frac{\xi}{|\xi|}\right).$$

Note that $\xi/|\xi|$ corresponds to the Riesz transform while $H(\xi)(X)$ is an odd mapping of ξ for each X, therefore $\int_{S^2} H(\xi)(X)\,d\sigma(\xi) = 0$. By [19, p. 75], we may conclude that the constant c is zero. Thus

$$(Tf)(x) = cf(x) + \text{p.v.} \int_{\mathbb{R}^n} K(x - y) f(y)\,dy$$

is bounded from $L^p(R^n, M^{3\times 3})$ to itself $(1 < p < +\infty)$ and satisfies the weak-(1, 1) estimate.

Given a subspace $E \subset M_s^3$ of incompatible linear strains, the operator T makes it possible for us to study the Beltrami-like system of first order equations in the form $P_{E^\perp}(e(Du(x))) = H(x, P_E(e(Du(x))))$, where $H : \mathbb{R}^n \times E \to E^\perp$ satisfies the Lipschitz condition $|H(x, X_1) - H(x, X_2)| \leqslant k|X_1 - X_2|$ for $X_1, X_2 \in E$. We can also consider invertibility and spectral properties in $L^p(\mathbb{R}^n, E^\perp)$ for, say the operator $P_{E^\perp}(T)$ (see [3]) for a modern treatment of the classical Beltrami operator S on the plane which satisfies $S(\bar{\partial} f) = \partial f$. We will consider such an extension elsewhere.

Now we use the following example to illustrate the effect of exposed faces of $C(K)$ without compatible connections on $Q^e(K)$.

Example 1. We first consider the three-strain configuration $K = \{A, B, C\}$ without compatible strains [7] in the subspace of 3×3 diagonal matrices. The line segments are compatible strains. It was shown in [7] that $L_c^e(K)$ is given by the three 'legs' and the small triangle thus formed (see Fig. 1). Now suppose we only know that $K \subset M_s^3$ is a three-point set without compatible strains, we would like to bound $Q^e(K)$ by using this very limited information. By Theorem 1, we see that $Q^e(K)$ is contained in the set given by Fig. 1.

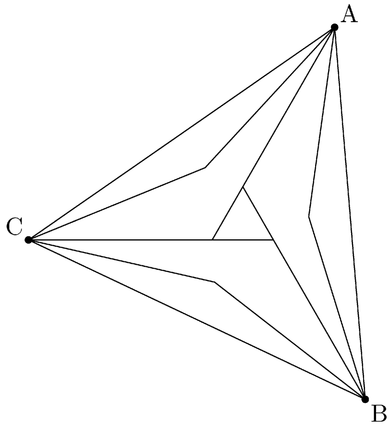

Fig. 1. The three-strain set, its lamination convex hull and the bound given by Theorem 1.

On the other hand, how much we can chip into the triangle ABC depends on the constant C_E given by (19) in the proof of Theorem 2.

Now we turn to approach (ii). The assumptions on K, L, M, W^*, V^*, e and F^* are the same as those defined just before Theorem 1. Since $L \subset C(K)$ is an exposed 1-face, there are at least two points $X_1, X_2 \in L \cap K$ and we assume that the open edge $\Gamma = (X_1, X_2) := \{X \in M_s^3,\ X = (1-t)X_1 + tX_2,\ 0 < t < 1\}$ does intersect with K, that is, $\Gamma \cap K = \emptyset$. This is possible because K is finite. If $L \cap C(K)$ contains more than two points, we give our estimates pair by pair as above. Let $X_0 \in \Gamma$ be the mid-point of Γ, $X_0 = (X_1 + X_2)/2$, we will show that the quasiconvex hull of K is contained in the sub-level set of a rank-one convex quadratic function defined on linear strains such that the intersection of K the boundary of the sub-level set can contain at most X_1 and X_2. From this we can see that a neighbourhood of X_0 is not in $Q^e(K)$.

We can translate the origin of M_s^3 to X_0 and we may assume, without loss of generality that $E = \mathrm{span}[L]$ is a one-dimensional linear subspace of M^{s3} without compatible strains and $X_0 = 0$ with E_0 a unit vector in E. Now we have $X_2 = -X_1$ and let $h := |X_1| = |X_2| > 0$.

Now according to (18), we see that the quadratic form on linear strains defined by

$$q_t(X) = \left|P_{E^\perp}\big(e(X)\big)\right|^2 - t\mu\left|P_E\big(e(X)\big)\right|^2, \quad X \in M^{3\times3},\ 0 \leqslant t \leqslant 1,\ \mu = 1 - \lambda_+^2 - \lambda_-^2, \tag{22}$$

is a rank-one convex quadratic form defined on $M^{3\times3}$.

Recall that we have defined the d-dimensional plane M (now a subspace) which is the subspace spanned by $C(K)$ and e is the unit normal vector of W pointing towards the inward side of $C(K)$. Now we consider the set $P_F(C(K)) \subset F$ where $F = \mathrm{span}[e] \oplus V$. The projection remains a convex set and $P_F(X_1) = P_F(X_2) = 0$ and 0 is an exposed point of the polytope $P_F(C(K))$. Now it is possible to find a closed ball $\bar{B}_R(Re) := \{Y \in F,\ \|Y - Re\| \leqslant R\} \subset F$ centred at a point te at least for some large $R > 0$ such that the ball contains $P_F(C(K))$ and the boundary sphere $\partial \bar{B}_R(te)$ intersects with $P_F(C(K))$ only at $0 = P_F(X_1) = P_F(X_2)$.

Next we consider the quadratic function

$$q_{t,s}(Y) = \left|P_{E^\perp}\big(Y - (R-s)e\big)\right|^2 - t\mu\left|P_E\big(Y - (R-s)e\big)\right|^2 = \left|P_{E^\perp}\big(Y - (R-s)e\big)\right|^2 - t\mu\left|P_E(Y)\right|^2$$

for $0 \leqslant s < R$ with s small. Now for large $R > 0$ and small $s > 0$ and $t > 0$, let us consider the closed sub-level set of the quadratic function $q_{t,s}$ defined by

$$U_{s,t} = \left\{Y \in M_s^3,\ q_{t,s}(Y) \leqslant (R-s)^2 - t\mu h^2\right\}$$

which is a quasiconvex set of linear strains, we see that $X_1, X_2 \in \partial U_{s,t}$ and for sufficiently small $t, s > 0$, we may claim that $K \subset U_{s,t}$. Also we see that $q_{t,s}(0) = (R-s)^2 > (R-s)^2 - t\mu h^2$ so that at a neighbourhood of 0 does not intersect with $U_{s,t}$. Thus we see that $Q^e(K)$ is contained in the domain bounded by a one sheet hyperboloid defined by $q_{t,s} = (R-s)^2 - t\mu h^2$.

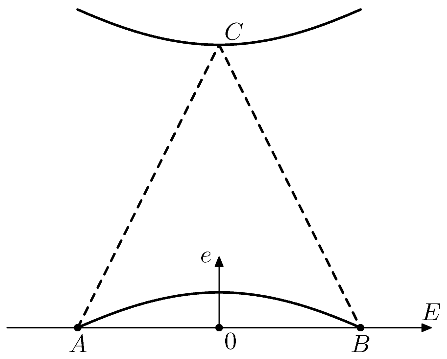

Fig. 2. An estimate of the quasiconvex hull near the incompatible line segment $[A, B]$.

A quick for deriving a bound is to consider the following family of sub-level sets starting with the domain bounded by the two-sheet hyperbola:

$$U_{t,h} = \{ |P_{E^\perp}(Y - te)|^2 - \mu |P_E(Y)|^2 \leqslant t^2 - \mu h^2,\ Y \in M,\ t \geqslant 0 \},$$

so that we always have $X_1, X_2 \in U_{t,h}$. By increasing $t > 0$ to $+\infty$, the one-sheet hyperbola $(y-t)^2 - \mu x^2 = t^2 - \mu h^2$ $(x, y) \in \mathbb{R}^2$ approaches the line $y = 0$ in any compact set of $x \in \mathbb{R}$ so that we can find a finite t which satisfies $K \subset U_{t,h}$ and the bound near the 1-face L is then obtained.

Remark 5. The construction of the above bound of quasiconvex hull can be optimised by considering the largest $t > 0$ such that $K \subset U_{s,t}$ holds.

We would also like to remark that similar to the method used in [2] for Beltrami operators, we can show, by using the operator T that homogeneous strain Young measures supported in a compact subset of the cone $C := \{Y \in M_s^3, |P_{E^\perp}(Y)| \leqslant \lambda |P_E(X)|\}$ can be generated by a sequence of strains $e(Du_j)$ whose image is contained in the cone C. We will present this type of results elsewhere.

Example 2. Let us consider an isosceles triangle in Fig. 2 with incompatible face $[A, B]$. We take the mid-point of the line segment as the original and $E = \mathrm{span}[B - A]$ with e the unit vector in $M = \mathrm{span}[A, B, C]$. Let assume that $A = (-2, 0)$, $B = (2, 0)$, $C = (0, 4)$ and $\mu = 1/2$. By taking $t = 2.25$, we can draw the level set of the quadratic function q and bound the quasiconvex hull of $K = \{A, B, C\}$ based on the incompatible face $[A, B]$ as below. The calculations are elementary and the bound is easily obtained. However, if we imagine that C moves upward along the e axis towards infinity, the lower part of the hyperbola will become flatter and flatter. Consequently, our bounds will be poor and poor. This shows that the bounds by rank-one convex quadratic functions are size dependent.

Acknowledgment

We wish to thank the referee for several very helpful suggestions.

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